Math 1A: Calculus **Instructor: Alexander Paulin**

Handout: Exploring the Power Rule for Derivatives

Discussions 201, 203 // 2018-10-01

The power rule for derivatives states that if $f(x) = x^n$, its derivative is given by

$$f'(x) = nx^{n-1}.$$

First, we make a few notes about this law. One of the most useful and immediate corollaries is that it tells us how to compute the derivative of any polynomial:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

(This is because the derivative of a sum is the sum of the derivatives.)

Actually, the power law holds even when n is not a whole number. Indeed, the rule is valid for *any* exponent n, whether it be $n = 3/2$ or $n = -1.57$ or $n = 0$...

Exercise 1. Check the last of these—namely, that the derivative of $f(x) = x^0$ is indeed given correctly by the power law.

There is a subtle point pertaining to domains that arises in using the exponent law with more exotic choices of n, however—a subtlety that arose in a homework problem on Homework 5.

The example you saw on the homework was $f(x) = x^{3/2}$, whose derivative according to the power law is $f'(x) = \frac{3}{2}x^{1/2}$. While this is true, it is slightly misleading for the following reason: the domain of f is $\underset{\text{interval}}{\underline{\hspace{2cm}}}$, the domain of $\frac{3}{2}x^{1/2}$ is $\underset{\text{interval}}{\underline{\hspace{2cm}}}$, but the domain of $f'(x)$ is only $\underset{\text{interval}}{\underline{\hspace{2cm}}}$. Thus it would be more accurate to say that $f'(x)$ is given by the power law—but only where it exists!

Exercise 2. Find all values of n for which the function $f(x) = x^n$ meets each of the following conditions:

(1) Both $f(x)$ and $f'(x)$ have domain $\mathbb{R}$.
(2) Both $f(x)$ and $f'(x)$ have domain $(-\infty, 0) \cup (0, \infty)$.
(3) $f(x)$ has domain $\mathbb{R}$ but $f'(x)$ has domain $(-\infty, 0) \cup (0, \infty)$.
(4) $f(x)$ has domain $[0, \infty)$ but $f'(x)$ has domain $(0, \infty)$.

Why is it impossible for both $f(x)$ and $f'(x)$ to have domain $[0, \infty)$?

The rest of this worksheet will be dedicated to exploring another phenomenon of the power law.

Exercise 3. Does there exist a function $f(x) = cx^n$, for some constants c and n, such that $f'(x) = x^{-1} = 1/x$?

Exercise 4. Compute the derivative of the function $f(x) = 1000x^{0.001} - 1000$. If you have a graphing calculator, graph $y = f(x)$ for x in the interval $[0, 10]$. Does this look like another function you've seen before?

Exercise 5. Define F_α (for a fixed number $\alpha > 0$) to be the function

$$F_\alpha(x) = \frac{1}{\alpha}(x^\alpha - 1).$$

For example, the function considered in the previous problem would be called $F_{0.001}$. Compute F'_α. What happens to F'_α as you let α approach 0?

Exercise 6. In the previous problem you were asked to think about what happens to F'_α as α approaches 0. In this problem, we'll consider what happens to F_α. That is to say, consider the function F defined as the "limit of the functions F_α":

$$F(x) = \lim_{\alpha \to 0^+} F_\alpha(x) = \lim_{\alpha \to 0^+} \frac{x^\alpha - 1}{\alpha}.$$

Write out the equation of $F(x)$. (In other words: compute the above limit in terms of x. Hint: we've done it before.)

It is in fact true that $F'(x) = 1/x$ (for $x > 0$), but what we've done so far doesn't actually constitute a "proof" of this fact. (An actual proof using implicit differentiation is actually quite short and can be found at the start of §3.6 in the text. We'll see it in lecture later.)